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INFERRING A LINEAR ORDERING OVER A POWER SET

ABSTRACT. An observer attempts to infer the unobserved ranking of two ideal objects, A and B, from observed rankings in which these objects are 'accompanied' by 'noise' components, C and D. In the first ranking, A is accompanied by C and B is accompanied by D, while in the second ranking, A is accompanied by D and B is accompanied by C. In both rankings, noisy-A is ranked above noisy-B. The observer infers that ideal-A is ranked above ideal-B. This commonly used inference rule is formalized for the case in which A, B, C, D are sets. Let X be a finite set and let $\succ$ be a linear ordering on 2^X. The following condition is imposed on $\succ$. For every quadruple $(A, B, C, D) \in Y$, where Y is some domain in $(2^X)^4$, if $A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C$, then $A \succ B$. The implications and interpretation of this condition for various domains Y are discussed.

KEY WORDS: Cross inferences, Inference rules, Prior knowledge, Ranking

1. INTRODUCTION

Decision makers often need to know the ranking of objects, yet cannot observe it directly. For example, an economist wants to rank two ideal policies by their effect on GDP, yet the policies she can observe are only non-ideal, politically constrained ones. In such cases, decision makers may try to guess the unobserved ranking on the basis of other comparisons they do get to observe. This paper studies an inference rule which is commonly used in such cases. Suppose that an observer is interested in the ranking of two objects, A and B. However, she can only observe comparisons in which the objects are 'accompanied' by other objects, C and D. In particular, she observes two comparisons: (1) A accompanied by C versus B accompanied by D; (2) A accompanied by D versus B accompanied by C. The observer sees that in both comparisons, the former 'noisy object' is ranked above the latter. She infers that in the ideal comparison between A and B, A would be ranked above B.

I study this inference rule for the special case in which A, B, C, D are all finite *sets*. Consider the following example. A prescription is

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a set of fitness-improving habits that a physician may recommend to a patient (e.g., eat broccoli, swim, do not smoke). The physician wishes to rank two prescriptions, A and B, according to their contribution to fitness. Initially, patient 1 is prescribed A and patient 2 is prescribed B, and patient 1 outperforms patient 2 in the fitness test. The physician decides to reverse the prescriptions: she prescribes B to patient 1 and A to patient 2. Consequently, patient 2 does better than patient 1 in the fitness test. The physician concludes that A is better than B.

The physician's inference employs an intuitive rule of thumb which is familiar from everyday experience. The rule can be expressed in set-theoretic terms. Let X be a set of fitness-improving habits. A subset of X is a (healthy) 'life-style', i.e., a collection of fitness-improving habits adopted by a patient at a given time, whether on prescription or not. Let C and D be the life styles conducted by patients 1 and 2 *independently of the physician's prescriptions*. Let $\succ$ be a linear ordering (i.e., a complete, irreflexive and transitive relation) over 2^X. This ordering represents the ranking of all life styles by their contribution to fitness. The physician's inference rule can thus be stated as follows: If $A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C$, then $A \succ B$. I refer to this rule as *cross inference*. The question I study in this paper is: *What is the class of linear orderings $\succ$ over 2^X, for which the rule is valid*, i.e., $[A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C]$ implies $A \succ B$?

The answer turns out to depend on further specifications of the inference procedure. Returning to our life-style example, note that it ignores the possibility of non-empty intersection between a prescription and the patients' independent life styles. For instance, the prescription A may include the recommendation to swim while patient 1's independent life style C already includes the habit of swimming, regardless of the physician's prescription (i.e., $A \cap C \neq \phi$). This is referred to as Type I intersection. Another type of non-empty intersection can occur between one patient's independent life style and the other's (i.e., $C \cap D \neq \phi$). For instance, the two patients may both be swimming, independently of the physician's prescription. This is referred to as Type II intersection.

To see how ignoring these details can lead to wrong inferences, consider the following example. Let $X = \{1, 2, 3\}$. Suppose that the true order $\succ$ on 2^X induces $\{2, 3\} \succ \{1, 2\} \succ \{1, 3\} \succ \{3\} \succ \{2\} \succ \{1\}$. It is easy to see that if the observer ignores Type II intersection, she will incorrectly conclude from the observation $\{1, 2\} \succ \{1, 3\}$ that $\{2\} \succ \{3\}$ (to see why, let $A = \{2\}$, $B = \{3\}$, $C = D = \{1\}$ and employ cross inference). As to Type I intersection, note that two of the rankings induced by $\succ$ can be rewritten as $\{1\} \cup \{2\} \succ \{3\} \cup \{3\}$ and $\{3\} \cup \{2\} \succ \{3\} \cup \{1\}$. Let $A = \{2\}$, $B = D = \{3\}$, $C = \{1\}$, employ cross inference and obtain the incorrect conclusion $\{2\} \succ \{3\}$. The intuition behind this error is simple: to ascribe an improvement in a patient's relative fitness to the prescription of a certain life style may be erroneous if the patient conducts this life style independently of the physician's prescription.

Ideally, a fully rational observer would be aware of these intersections as well as their implications on the validity of cross inference. Conversely, a boundedly rational observer may instinctively employ cross inference because of the rule's intuitive appeal and familiarity, without being aware of the intersections or their exact implications. Alternatively, the observer may find it impossible or too costly to control for the intersections. In our life-style example, the physician may be limited, prior to giving the patients her prescriptions, to asking them only one yes/no question, such as: (a) 'Do you, patient 1, already have any of the habits on the prescription'? (b) 'Do you, patient 1, have any of patient 2's habits'? Question (a) identifies only Type I intersection while question (b) identifies only Type II intersection. Finally, the observer may not have any better inference method at hand than cross inference. It is therefore interesting to know under what conditions cross inference is logically valid, for the four possible cases: both intersection types / only Type I intersection / only Type II intersection / none of the intersection types are allowed.

The results can be summarized as follows. When only Type I intersection is allowed (i.e., when cross inference is required to hold for all $A, B, C, D \subset X$ such that $C \cap D =\succ \phi$), cross inference is valid only for max–max orders (i.e., $A \succ B$ whenever $\{x\} \succ \{y\}$ for some $x \in A$ and every $y \in B$). When only type II intersection is allowed (i.e., when cross inference is required to hold for all $A, B, C, D \subset X$ such that $(C \cup D) \cap (A \cup B) = \phi$), cross inference is valid if and only if $\succ$ satisfies the following *independence* property: for every $A, B, C \subset X$ such that $C \cap (A \cup B) = \phi$, $A \succ B$ implies $A \cup C \succ B \cup C$. When both Type I and Type II intersections are allowed (i.e., when cross inference is required to hold for all A, B, C, $D \subset X$), there exists no order on 2^X for which cross inference is valid. When both Type I and Type II intersections are *not* allowed (i.e., when cross inference is required to hold for all $A, B, C, D \subset X$ such that $C \cap D = (C \cup D) \cap (A \cup B) = \phi$), cross inference is valid for a wide class of orders (e.g., every order that satisfies the above independence property, but also some which violate it), which I am unable to characterize formally.

The discussion so far implicitly assumed that the observer has no prior knowledge about the ordering she observes. In our life-style example, however, the physician may know that the effect of life style on fitness satisfies a 'between-ness' property: whenever $A \cap B = \phi$, $A \succ B$ implies $A \succ A \cup B \succ B$. The interpretation is that combining a bad-quality life style with a completely different, good-quality life style yields a medium-quality life style. Alternatively, the physician may know that $\succ$ satisfies a 'monotonicity' property: $A \supset B$ implies $A \succ B$, the interpretation being that adding fitness-improving habit to any life style increases fitness. Such partial prior knowledge may narrow further the class of orderings for which cross inference is a valid procedure. This idea is illustrated in the final section of the paper. In particular, it is shown that imposing properties such as between-ness or monotonicity, can imply that cross inference is valid for a *unique* order.

Related literature

The formalism employed in this paper is essentially borrowed from the literature on the problem of extending an order on a set to its power set. In this literature, an ordering $>$ is imposed over a set X and an ordering $>^*$ is imposed over the power set 2^X, such that $>^*$ coincides with $>$ over singletons. The problem is to characterize $>^*$ in terms of $>$ (e.g., $>^*$ lexicographic with respect to $>$). This formalism has various standard interpretations. One of the original motivations for studying this problem was social choice theoretic. The problem was to define voters' preferences when social choice correspondences, rather than functions, are implemented (see Pattanaik and Peleg, 1984). A different interpretation is grounded in dynamic decision theory: $>^*$ represents a decision maker's first-stage choice in a two-stage dynamic choice problem. At the first stage, she chooses a menu and at the second stage, she chooses an element from it. Properties of $>^*$ represent dynamic-choice considerations: monotonicity represents the decision maker's preference for flexibility due to her anticipation of unforeseen contingencies, which will change her future tastes (Kreps, 1979; Dekel et al., 2000); between-ness represents preference for commitment due to anticipated self-control problems (Gul and Pesendorfer, 2000). The formalism is also used in welfare economics, where the motivation is to formulate criteria for evaluating an agent's freedom of choice, based on the agent's opportunity set (Sen, 1991; Puppe, 1995).

The present paper thus provides a new interpretation for the formalism of a linear ordering over a power set. Properties of orders on 2^X represent inference rules used by an observer, whose objective is to infer unobserved rankings from observed ones. Note a slight difference between the order-extension formalism and the present model, where there is no need to assume a primitive order on X and it is therefore dropped.

The remainder of the paper is structured as follows. Section 2 introduces cross-inference. Section 3 presents the main results. Section 4 analyzes cross inference with partial prior knowledge.

2. CROSS INFERENCE

Let $X = \{1, 2, \ldots, n\}$ $(n > 2)$ and let 2^X denote its power set. Endow 2^X with a linear ordering $\succ$ (i.e., a complete, irreflexive and transitive relation), which induces the ranking $\{n\} \succ \{n-1\} \succ \cdots \succ \{1\}$ over singletons. The sets X and 2^X can be given various interpretations: (1) X is a set of alternatives and 2^X is the set of all opportunity sets. (2) X is a set of properties and every element in 2^X corresponds to some object, such that distinct objects are characterized by distinct subsets of properties. For instance, candidates for a vacancy are 'objects' and each candidate is profiled by a set of binary characteristics (technical education, experience in marketing, etc.). (3) X is a set of individual elements and 2^X is the set of all 'teams'. The life-style example of Section 1 fits this interpretation. Under interpretation (1), $\succ$ represents preferences over opportunity sets. Under interpretations (2) and (3), $\succ$ represents the relative quality of objects/teams.

I impose one main axiom on $\succ$. In its general form, the axiom requires that for all $(A, B, C, D) \in Y$ (where Y is some domain in $(2^X)^4$), if $A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C$, then $A \succ B$. This condition is labeled *'cross inference'*. I refer to A and B as *'primary sets'* and to C and D as *'auxiliary sets'*. Cross inference is interpreted as an inference rule used by an observer in order to extract information about $\succ$ from a sample of observed rankings. The observer lacks direct access to the ranking between the primary sets and wishes to infer it from two observed rankings: $A \cup C$ vs. $B \cup D$ and $A \cup D$ vs. $B \cup C$. The two auxiliary sets, C and D, can be viewed as 'noise' components. In the first comparison, A is 'accompanied' by C and B is accompanied by D. In the second comparison, A is accompanied by D and B is accompanied by C. Cross inference means that if accompanied-A is ranked above accompanied-B in both comparisons, the observer infers that A is ranked above B.

This is an intuitive and familiar rule of thumb from everyday inferences. Our objective will be to find conditions under which using the rule is logically valid. Formally, for a given domain (of quadruples of sets) Y, the problem is to find the class of linear orderings $\succ$ on 2^X satisfying that for all $(A, B, C, D) \in Y$, $[A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C]$ implies $A \succ B$. Four domains are considered:

Case 1: $Y = \{(A, B, C, D) \in (2^X)^4; C \cap D = \phi\}$. In this case, cross inference is required to hold only when the auxiliary sets are mutually disjoint. To borrow the terminology of Section 1, only Type I intersection is allowed. The cross inference condition can thus be stated as follows:

CI(1): For all $A, B, C, D \subseteq X$ such that $C \cap D = \phi$, if $A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C$, then $A \succ B$.

Case 2: $Y = \{(A, B, C, D) \in (2^X)^4; (C \cup D) \cap (A \cup B) = \phi\}$. In this case, cross inference is required to hold only when auxiliary sets do not intersect primary sets. To put it differently, only Type II intersection is allowed. Cross inference can thus be stated as follows:

CI(2): For all $A, B, C, D \subseteq X$ such that $(C \cup D) \cap (A \cup B) = \phi$, if $A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C$, then $A \succ B$.

Case 3: $Y = (2^X)^4$. There are no domain restrictions in this case. That is, both Type I and Type II intersections are allowed. Cross inference can thus be stated as follows:

CI(3): For all $A, B, C, D \subseteq X$, if $A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C$, then $A \succ B$.

Case 4: $Y = \{(A, B, C, D) \in (2^X)^4; C \cap D = (C \cup D) \cap (A \cup B) = \phi\}$. In this case, cross inference is required to hold only when auxiliary sets intersect neither primary sets nor themselves. That is, neither of the two intersection types is allowed. Cross inference can thus be stated as follows:

CI(4): For all $A, B, C, D \subseteq X$ such that $C \cap D = (C \cup D) \cap (A \cup B) = \phi$, if $A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C$, then $A \succ B$.

Thus, we have four different versions of cross inference, each defined over a different domain of quadruples of sets. This categorization represents natural constraints on the inference procedure. As we saw in the life-style example of Section 1, it is possible for the observer to ignore Type I or Type II intersections (or fail to understand their implications), while continuing to rely on cross inference. Alternatively, the observer may find it impossible or too costly to control for the intersection types. Finally, the observer may have no better inference method at hand and may therefore employ cross inference because of its familiarity and intuitive appeal.

A special case of cross inference that will play a role in one of the main results involves empty primary or auxiliary sets. In terms of the life-style example of Section 1, an empty primary set corresponds to comparing giving a single prescription with giving no prescription at all. Thus, if a patient under prescription always performs worse than a patient without any prescription, the physician's intuitive inference is that giving the prescription is worse than giving no prescription at all. In terms of the same example, an empty auxiliary set corresponds to a patient with no independent fitness-enhancing habits.

3. MAIN RESULTS

What follows is an analysis of cases 1–4. For each case, I identify the class of linear orderings for which the relevant version of cross inference is valid. Occasionally, I also make the natural assumption that every non-empty set is ranked above the empty set:

Desirability (D): $A \neq \phi$ implies $A \succ \phi$.

Case 1. Recall that in this case, cross inference is required to hold as long as auxiliary sets do not intersect each other. The following result shows that the implications of CI(1) are quite strong.

DEFINITION. For all $A \subseteq X$, let $k = \max(A)$ if $\{k\} \succ \{j\}$ for every $j \in A$, $j \neq k$. An ordering $\succ$ is max–max if for all $A, B \subseteq X$, $\{\max(A)\} \succ \{\max(B)\}$ implies $A \succ B$.

PROPOSITION 3.1. *If $\succ$ satisfies CI(1) and D, then $\succ$ is max–max.*

Proof. Let us first prove three auxiliary claims. Let $A, B, C \subset X$ be pairwise disjoint:

LEMMA 1. *If $A \cup B \succ C$, then $B \succ C$ or $A \succ C$.*

Proof. $A \cup B \succ C \cup C$. Now, either $A \cup C \succ B \cup C$ or $B \cup C \succ A \cup C$. By CI(1), $A \succ C$ in the former case and $B \succ C$ in the latter.

LEMMA 2. *If $A \succ B$ and $A \succ C$, then: (i) $A \succ B \cup C$; (ii) $A \cup B \succ C$.*

Proof. (i) Assume that $B \cup C \succ A$. By Lemma 1, $B \succ A$ or $C \succ A$, a contradiction. (ii) Assume that $C \succ A \cup B$. This statement can be rewritten as $\phi \cup C \succ B \cup A$. By (i), $A \succ B \cup C$, which can be rewritten as $\phi \cup A \succ B \cup C$. By CI(1), $\phi \succ B$, a contradiction.

LEMMA 3. *If $A \succ B$ and $A \succ C$, then $A \cup C \succ B \cup C$.*

Proof. Assume, without loss of generality, that $B \succ C$. Now, either (i) $B \cup C \succ B \cup A$, or (ii) $B \cup A \succ B \cup C$. If (i) holds and $B \cup C \succ A \cup C$, then we can rewrite these statements as $B \cup (B \cup C) \succ A \cup C$ and $C \cup (B \cup C) \succ A \cup B$. By CI(1), $B \cup C \succ A$, which contradicts Lemma 2(i). If, on the other hand, (ii) holds and $B \cup C \succ A \cup C$, then by transitivity, $B \cup A \succ A \cup C$. Rewrite these statements as $A \cup B \succ (A \cup C) \cup C$ and $C \cup B \succ (A \cup C) \cup A$. By CI(1), $B \succ A \cup C$, in contradiction to Lemma 2(ii).

Now, let us show that for all $A, B \subseteq X$, if $\{\max(A)\} \succ \{max(B)\}$, then $A \succ B$. First, consider the case of $A \cap B = \phi$. Applying Lemma 2(i) inductively yields $\{\max(A)\} \succ D$ for all $D \subseteq \{1, \ldots, \max(A) - 1\}$. Therefore, $\{\max(A)\} \succ B$ and $\{\max(A)\} \succ A - \{\max(A)\}$. Thus, by Lemma 2(ii), $A \succ B$. Now suppose $A \cap B \neq \phi$. Denote $C = A \cap B$, $A' = A - C$ and $B' = B - C$. Since A', B', C are pairwise disjoint and $B', C \subseteq \{1, 2, \ldots, \max(A) - 1\}$, $A' \succ C$ and $A' \succ B'$. Thus, by Lemma 3, $A \succ B$. $\square$

Thus, for case 1, cross inference is valid only for max-max orders.[1] To illustrate the logic of this result, let us return to the life-style example of Section 1. Let X = {*swimming, eating broccoli, no smoking*}, such that {*swimming*} $\succ$ {*no smoking*} $\succ$ {*eating broccoli*}. Suppose that the physician wants to infer the ranking between swimming and eating broccoli. She first prescribes swimming to patient 1 and broccoli to patient 2, ignoring the fact that patient 1's and 2's independent healthy life styles consist of swimming and no smoking, respectively. Suppose that contrary to Proposition 3.1, patient 2 outperforms patient 1 in the fitness test: {*no smoking*} $\cup$ {*eating broccoli*} $\succ$ {*swimming*} $\cup$ {*swimming*}. The physician reverses the prescriptions — swimming is prescribed to patient 2 and broccoli is prescribed to patient 1. Suppose that now patient 1 does better in the fitness test: {*swimming*} $\cup$ {*no smoking*} $\succ$ {*swimming*} $\cup$ {*eating broccoli*}. By CI(1), the physician infers that {*no smoking*} $\succ$ {*swimming*}, a contradiction. A similar contradiction can be obtained if {*swimming*} $\cup$ {*eating broccoli*} $\succ$ {*swimming*} $\cup$ {*no smoking*}.

Case 2. In this case, cross inference holds when auxiliary sets intersect each other but not when they intersect primary sets.

DEFINITION. An ordering $\succ$ satisfies *independence* if $A \succ B$ implies $A \cup C \succ B \cup C$ whenever $C \cap (A \cup B) = \phi$.

PROPOSITION 3.2. *$\succ$ satisfies CI(2) if and only if $\succ$ satisfies independence.*

Proof. It is trivial to see that CI(2) implies independence. Let $C = D$ and $C \cap (A \cup B) = \phi$. Suppose that $B \succ A$ and $A \cup C \succ B \cup C$. The latter can be rewritten as $A \cup C \succ B \cup D$ or $A \cup D \succ B \cup C$. By CI(2), $A \succ B$, a contradiction. Now, let us show that independence implies CI(2). Let $A, B, C, D \subset X$ satisfy $C \cap (A \cup B) = D \cap (A \cup B) = \phi$. Suppose that $A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C$ and yet $B \succ A$. Independence implies $B \cup C \succ A \cup C$. By transitivity, $A \cup D \succ B \cup C \succ A \cup C \succ B \cup D$ and thus, $A \cup D \succ B \cup D$. But by independence, it follows from $B \succ A$ that $B \cup D \succ A \cup D$, a contradiction. □

Thus, if cross inference is applied whenever auxiliary and primary sets do not intersect, it turns out to be equivalent to independence. This property and its variants have been well-studied in the literature (e.g., Barbera et al., 1984; Kannai and Peleg, 1984; Bossert et al., 1994). Various natural orders satisfy independence: additive orders ($A \succ B$ if $\Sigma_{x \in A} f(x) > \Sigma_{y \in B} f(y)$ for some $f : X \to N$); the regular lexicographic order; orders which combine size-dependence and the additive order (e.g., $A \succ B$ if and only if $\mid A \mid > \mid B \mid$, or $\mid A \mid = \mid B \mid$ and $\Sigma_{x \in A} f(x) > \Sigma_{y \in B} f(y)$); etc. If the observer knows that the order she tries to discover satisfies independence, then cross inference is valid, as long as she does not follow the rule when auxiliary and primary sets intersect. It should be noted that a similar, weaker relation between cross inference and independence exists in case 1 as well. Lemma 3 in the proof of Proposition 3.1 shows that CI(1) implies a weak form of independence: $A \succ B$ and $A \succ C$ imply $A \cup C \succ B \cup C$ whenever $C \cap (A \cup B) = \phi$.

Case 3. In this case, the lack of any domain restriction leads to an impossibility result:

PROPOSITION 3.3. *There exists no linear ordering on* 2^X *which satisfies CI(3).*

Proof. Let $X = \{1, 2, \ldots, n\}$ and $k, j, i \in X$ satisfy $\{k\} \succ \{j\} \succ \{i\}$. Recall that $n > 2$. Clearly, CI(3) implies CI(1), so that Proposition 3.1 holds. Suppose that $\{j, k\} \succ \{i, k\}$. This can be rewritten as $\{j\} \cup \{k\} \succ \{i, k\} \cup \{k\}$. By CI(3), $\{j\} \succ \{i, k\}$, in contradiction to Proposition 3.1. Now suppose that $\{i, k\} \succ \{j, k\}$.

This can be rewritten as $\{i\} \cup \{k\} \succ \{j\} \cup \{k\}$. By CI(3), $\{i\} \succ \{k\}$, a contradiction. □

Thus, the observer's failure to control for any of the two intersection types implies that cross inference is not generally valid.

Case 4. In this case, inference is required to hold only when neither of the two intersection types occurs. I am unable to provide a formal characterization result for this case, but a few examples can demonstrate that CI(4) is valid for a rich variety of orders on 2^X. By the reasoning of Proposition 3.2, cross inference (in the version CI(4)) is valid whenever $\succ$ satisfies independence. However, there exist orders which violate independence and yet satisfy CI(4), as the following example illustrates.

Let $X = \{1, 2, 3, 4\}$. Define a strict order $\succ'$ over a subset of 2^X, $\mathfrak{I} = \{\{4\}, \{3, 1\}, \{3\}, \{2\}, \{1\}\}$, such that $\{4\} \succ' \{3, 1\} \succ' \{3\} \succ' \{2\} \succ' \{1\}$. Now define a lexicographic order over 2^X with respect to $\succ'$: $A \succ B$ if $4 \in A$ and $4 \notin B$, or if $4 \in A, B$ and $\{3, 1\} \subseteq A$ and $\{3, 1\} \not\subseteq B$, and so forth. Thus, $\{4, 3\} \succ \{4, 1\}, \{4, 2\} \succ \{4, 1\}, \{4, 3, 1\} \succ \{4, 3, 2\}$ and $\{4, 3, 2\} \succ \{4, 2, 1\}$. This order satisfies CI(4) and D but violates independence. The interpretation for $\succ$ is that the pair $\{3, 1\}$ creates a 'positive externality', such that a set containing $\{1, 3\}$ may be ranked above a set which contains only 1 or 3.

4. CROSS INFERENCE WITH PARTIAL PRIOR KNOWLEDGE ABOUT THE ORDERING

The previous section implicitly assumed that the observer has no prior knowledge about the ordering when drawing inferences. In this section I extend the previous section's inquiry to the case in which the observer *does* have some prior knowledge about the ordering. I will focus on case 1, in which the version of cross inference is CI(1). Proposition 3.1 establishes a necessary condition for the validity of CI(1) but it does not establish sufficient conditions. Therefore, the observer's partial prior knowledge about the ordering may affect the validity of her inferences.[2] What follows is a list of natural properties, which the observer may know to be satisfied by $\succ$.

Monotonicity w.r.t. set inclusion (MON). For all $A, B \neq \phi$, if $A \supset B$, then $A \succ B$.
Between-ness (BET). For all $A, B \neq \phi$, $A \cap B = \phi$, if $A \succ B$, then $A \succ A \cup B \succ B$.

MON means that a non-empty set is better than any of its non-empty subsets. For instance, adding agents to a team makes it better, or adding alternatives to an opportunity set makes it more desirable (as in Kreps, 1979). BET, on the other hand, means that the union of two disjoint sets is better than its worse component and worse than its better component. For instance, adding ex-ante worse, 'tempting' elements to an opportunity set yields a less desirable set than the original set but a more desirable set than a set consisting of just the temptation itself (as in Gul and Pesendorfer, 2000).

The final property I consider allows for a natural form of violation of independence. The observer may not be sure that $\succ$ satisfies independence. However, she may know that independence can only be violated because of an interaction between some *pair* of elements. The following notation will be useful. For all $A \subset X$, $k \notin A$, define $A_{j\to k}$ as follows: if $j \in A$, then $A_{j\to k} = A - \{j\} \cup \{k\}$; if $j \notin A$, then $A_{j\to k} = A$. Thus, $A_{j\to k}$ is simply the set A after replacing the element j with the element k.

Pairwise dependence (PD). Let $A - B \neq \phi$ and $B - A \neq \phi$. Suppose that $A \succ B$ and $B \cup \{k\} \succ A \cup \{k\}$ for some $k \notin A \cup B$. Then, there exists $j \in (A - B) \cup (B - A)$ such that $A_{j\to k} \succ B_{j\to k}$ and $B_{j\to k} \cup \{j\} \succ A_{j\to k} \cup \{j\}$.

This property means that a pair of elements is identified, $\{j, k\}$, whose containment in a set may cause independence to be violated. This is a weaker condition than independence and to my knowledge, it is a first formalization in the literature of any notion of systematic dependence.

The following example illustrates PD. Let $X = \{1, 2, 3, 4\}$ and $\{4\} \succ \{3\} \succ \{2\} \succ \{1\}$. Suppose that $\{4, 2\} \succ \{4, 1\}$ but $\{4, 1, 3\} \succ \{4, 2, 3\}$. One way to interpret these comparisons is that the pair $\{2, 3\}$ creates a negative externality. PD implies that if the roles of 2 and 3 are exchanged, the rankings $\{4, 3\} \succ \{4, 1\}$ and $\{4, 2, 1\} \succ \{4, 2, 3\}$ must be obtained. That is, if the pair $\{2, 3\}$ weakens the right-hand set when the left-hand set includes 3, then it must do the same when 2 replaces 3 in the left-hand set.

How does the observer's prior knowledge that $\succ$ satisfies properties such as BET, MON or PD, affect the class of orderings for which CI(1) is valid? Before stating the characterization results, let us write down some definitions.

DEFINITIONS

(1) The ordering $\succ$ is *lex-max-max* if for all distinct $A, B \subseteq X$, if $k \in A$,$k \notin B$ and $A \cap \{k+1, \ldots, n\} = B \cap \{k+1, \ldots, n\}$, then $A \succ B$.
(2) The ordering $\succ$ is *lex-max-min* if for all distinct $A, B \subseteq X$, if $k \notin A$, $k \in B$, $A \cap \{1, \ldots, k-1\} = B \cap \{1, \ldots, k-1\}$ and there exists $k' \in A$ such that $\{k'\} \succ \{k\}$, then $A \succ B$.
(3) The ordering $\succ$ is *lex (max-max, lex-max-min)* if for all distinct $A, B \subseteq X$, $\{\max(A)\} \succ \{\max(B)\}$ implies $A \succ B$, and $\max(A) = \max(B)$ implies that $\succ$ follows the lex-max-min rule.

The lex-max-max order (which is the regular lexicographic order) and lex-max-min rules have been given several different derivations in the relevant literature (e.g., Pattanaik and Peleg, 1984; Bossert et al., 1994). The former (latter) rule ranks sets according to their lexicographically maximal (maxi-minimal) elements.[3]

To my knowledge, the third and more elaborate lex (max-max, lex-max-min) rule is new to the literature. This rule is illustrated by the following example, where $\succ$ represents the preferences of a decision maker (not the observer) over menus. Consider a restaurant diner who faces a choice between multi-course meals and employs the following choice procedure. He first compares the best dish in every meal. If the two are identical, he moves on to check the worst dish in every meal; he does so lexicographically, so that the meal with the better lexicographically minimal dish is chosen.

PROPOSITION 4.1. *$\succ$ satisfies assumptions CI(1), D, PD and MON if and only if it is lex-max-max.*

PROPOSITION 4.2. *$\succ$ satisfies assumptions CI(1), D, PD and BET if and only if it is lex (max-max, lex-max-min).*

Thus, in Proposition 4.1 (4.2), the prior knowledge that the ordering satisfies MON (BET) and PD leads to pinning down a unique ordering for which CI(1) is valid. Proofs are deferred to the end of this section. It remains to be checked whether any of the axioms imposed in this section are superfluous. Constructing extensions which satisfy D, PD and MON/BET but fail to satisfy CI(1) is fairly straightforward. An additive rule ($A \succ B$ if and only if $\sum_{x \in A} f(x) > \sum_{y \in B} f(y)$, for some $f : X \to N$) which does not induce the lex-max-max rule, satisfies assumptions D, PD and MON. An 'averaging' rule ($A \succ B$ if and only if $\sum_{i \in A-B} m(i)/ \mid A - B \mid > \sum_{j \in B-A} m(j)/ \mid B - A \mid$ in the case of $B - A \neq \phi$, and $\sum_{i \in A-B} m(i)/ \mid A - B \mid > \sum_{j \in B} m(j)/ \mid B \mid$ in the case of $B - A = \phi$, where $m : X \to N$ is properly defined) satisfies D, PD and BET. Both orders violate CI. As to orders which satisfy axioms CI, D and PD but violate MON or BET, these are of course explicitly derived in Propositions 4.1 and 4.2. Axiom D is implied by MON, but otherwise it is independent of the other axioms. An example for an order that satisfies CI(1), D and MON, yet violates PD, is given at the end of Section 3. A similar order which satisfies BET instead of MON can be constructed.

The proofs of Propositions 4.1 and 4.2 are now given. For the sake of readability, I substitute $k > j$ for $\{k\} \succ \{j\}$ for every $k, j \in X$, recalling that $X = \{1, 2, \ldots, n\}$ and that $\{n\} \succ \cdots \succ \{1\}$.

Proof of Proposition 4.1. First, I show that CI(1), D, PD and MON imply that $\succ$ is lex-max-max.
Step 1: By Proposition 3.1, $A \succ B$ when $\max(A) > \max(B)$.
Step 2: For every $k, i, j \in X$, $i > j$ implies $\{k, i\} \succ \{k, j\}$.
Proof. The case of $i > k$ follows from the lemma. Consider the case of $k > i$ and suppose that $\{k, j\} \succ \{k, i\}$. Then, by PD, either $i > k$ and $\{k, j\} \succ \{i, j\}$, or $k > j$ and $\{i, j\} \succ \{i, k\}$. We obtain a contradiction.
Step 3: $\{k, j\} \succ \{k, j - 1, \ldots, 1\}$, for all $k > j > 1$.
Proof. The proof is inductive. By step 2, $\{k, 2\} \succ \{k, 1\}$ for all $k > 2$. Assume that there exist $k > 3$ and $k - 1 > n$ such that $\{k, n\} \succ \{k, n - 1, \ldots, 1\}$ but $\{k, n, n - 1, \ldots, 1\} \succ \{k, n + 1\}$. Now, either (i) $\{k, n\} \cup \{n+1\} \succ \{k, n-1, ...1\} \cup \{n+1\}$, or (ii) $\{k, n-1, \ldots, 1\} \cup \{n + 1\} \succ \{k, n\} \cup \{n + 1\}$. If (i) holds, then we can write $\{n -$

$1, \dots, 1\} \cup \{k, n\} \succ \{k, n+1\} \cup \{n+1\}$ and $\{n+1\} \cup \{k, n\} \succ \{k, n+1\} \cup \{n-1, \dots, 1\}$. By CI(1), $\{k, n\} \succ \{k, n+1\}$, contradicting step 2. If (ii) holds, we can write $\{n+1\} \cup \{k, n-1, \dots, 1\} \succ \{k, n+1\} \cup \{n\}$ and $\{n\} \cup \{k, n-1, \dots, 1\} \succ \{k, n+1\} \cup \{n+1\}$. By CI(1), $\{k, n-1, \dots, 1\} \succ \{k, n+1\}$. By step 2, $\{k, n+1\} \succ \{k, n\}$. By transitivity, $\{k, n-1, \dots, 1\} \succ \{k, n\}$, a contradiction.
Step 4: $\succ$ is lex-max-max.
Proof. The proof is inductive. The lemma established that $\max(A) > \max(B)$ implies $A \succ B$. Now, let $\max(A) = \max(B) = h$ and suppose that $\max(A - \{h\}) > \max(B - \{h\})$. By step 3, $\{h, \max(A - \{h\})\} \succ \{h, \max(A - \{h\}) - 1, \dots, 1\}$. The set A can be generated from the left-hand side of this statement by adding elements, whereas B can be generated from the right-hand side of the statement by subtracting elements. Therefore, by MON and transitivity, $A \succ B$.

Now assume that $\succ$ is lex-max-max to the mth order (i.e., if the $m-1$ maximal elements in A and B are identical but the mth maximal element in A is ordered above the mth maximal element in B, then $A \succ B$). Assume, however, that there exist $A, B \subseteq X$ having identical m maximal elements, such that the $(m+1)$th maximal element in A is ordered above the *(m+1)*th maximal element in B, yet $B \succ A$. Denote by k the mth maximal element in A and B. By the inductive step, $A - k \succ B - k$. Note that $(A-k) - (B-k) = A - B$, $(B - \{k\}) - (A - \{k\}) = B - A$. By assumption, $A \not\subset B$. On the other hand, $A \supset B$ and $B \succ A$ contradict MON. Thus, both $A - B$ and $B - A$ are non-empty. By PD, there exists $j \in (A-B) \cup (B-A)$ such that: (i) $(A - \{k\})_{j \to k} \succ (B - \{k\})_{j \to k}$; (ii) $(B - \{k\})_{j \to k} \cup \{j\} \succ (A - \{k\})_{j \to k} \cup \{j\}$. Note that by the definition of k, $k > j$. If $j \in B$, then $B - \{k\})_{j \to k} = B - \{j\}$ and $A - \{k\})_{j \to k} = A - \{k\}$, implying that (i) contradicts the inductive step. It follows that $j \in A$. Hence, (ii) can be written as $\{j\} \cup (B - \{k\}) \succ A \cup \{k\}$ and the $B \succ A$ can be written as $\{k\} \cup (B - \{k\}) \succ A \cup \{j\}$. By CI(1), we have $B - \{k\} \succ A$, contradicting the inductive step. This establishes that $\succ$ is lex-max-max.

Now, let us show that assumptions CI(1), D, PD and MON are necessary for $\succ$ to be lex-max-max. The last three assumptions can easily be shown to be satisfied (in particular, PD is satisfied because lex-max-max satisfies independence). Let us prove that $\succ$ satisfies

CI(1). Let $C \cap D = \phi$. If $\max(A \cup C) \geqslant \max(B \cup D)$ and $\max(A \cup D) \geqslant \max(B \cup C)$, with at least one strict relation, then $\max(A) > \max(B)$. If $\max(A \cup C) = \max(B \cup D) = k$ and $\max(A \cup D) = \max(B \cup C) = k'$, then $k = k'$ and $\max(A) = \max(B) = k$, so we have to move on to the 2nd maximal elements. The same method of proof can be extended to the lexicographically maximal elements. □

Proof of Proposition 4.2. First, I show that CI(1), D, PD and BET imply that $\succ$ is lex (max-max, lex-max-min).
Step 1: By Proposition 3.1, $A \succ B$ when $\max(A) > \max(B)$.
Step 2: For all $j = 1, \ldots, n-2$ and all $k > j$, $\{k, k-1, \ldots, j+1\} \succ \{k, j\}$.
Proof. The proof is by induction on $k - j$. By step 2 in the proof of theorem 1, $\{k, j+1\} \succ \{k, j\}$, covering the case of $k - j = 2$. Assume that for some $k - m > 2$, $\{k, \ldots, k-m+1\} \succ \{k, k-m\}$ but $\{k, k-m-1\} \succ \{k, \ldots, k-m\}$. Now, either (i) $\{k, \ldots, k-m+1\} \cup \{k-m-1\} \succ \{k, k-m\} \cup \{k-m-1\}$ or (ii) $\{k, k-m\} \cup \{k-m-1\} \succ \{k, \ldots, k-m+1\} \cup \{k-m-1\}$. If (i) holds, we can write $\{k-m-1\} \cup \{k, k-m-1\} \succ \{k, k-m\} \cup \{k-1, \ldots, k-m+1\}$ and $\{k-1, \ldots, k-m+1\} \cup \{k, k-m-1\} \succ \{k, k-m\} \cup \{k-m-1\}$. By CI(1), $\{k, k-m-1\} \succ \{k, k-m\}$, in contradiction to step 2 in the proof of theorem 1. If (ii) holds, then we can write $\{k-m\} \cup \{k, k-m-1\} \succ \{k, \ldots, k-m+1\} \cup \{k-m-1\}$ and $\{k-m-1\} \cup \{k, k-m-1\} \succ \{k, \ldots, k-m+1\} \cup \{k-m\}$. By CI(1), $\{k, k-m-1\} \succ \{k, \ldots, k-m+1\}$. By step 2 in the proof of theorem 1, $\{k, k-m\} \succ \{k, k-m-1\}$. By transitivity, $\{k, k-m\} \succ \{k, \ldots, k-m+1\}$, contradicting the inductive step.
Step 3: $\succ$ is lex (max-max, lex-max-min).
Proof. The proof is inductive. Let $g = \min(A)$ if for every $i \in A$, $i > g$. Suppose that $\max(A) = \max(B)$, $\min(A) > \min(B)$. Denote $h = \max(A) = \max(B)$. By step 2, $\{h, h-1, \ldots, (\min(B)+1)\} \succ \{h, \min(B)\}$. Now, we can generate A by subtracting elements from the left-hand side of this statement, all of which are inferior to h. Likewise, we can generate B by adding elements to the right-hand side of this statement, all of which are inferior to h. Therefore, by BET and transitivity, $A \succ B$.

Assume that $\succ$ follows lex (max-max, lex-max-min) to the m-th order. I.e., if $\max(A) = \max(B)$, the $m-1$ minimal elements in A and B are identical, but the m-th minimal element in A is ordered above the m-th minimal element in B, then $A \succ B$. Assume, however, that there exist $A, B \subset X$ whose maximal elements and m minimal elements are identical, such that the $(m+1)$th minimal element in A is ordered above the $(m+1)$th minimal element in B, yet $B \succ A$. Denote by k the m-th minimal element in A and B. By the inductive step, $A - \{k\} \succ B - \{k\}$. Because A and B have identical maximal elements and the maxi-minimal element in A is ordered above the maxi-minimal element in B, it must be that $B \not\subset A$. On the other hand, if $A \subset B$, then BET is violated because $B \succ A$ and $A \succ B - A$ (the latter follows from the fact that $\max(A) > \max(B - A)$). Thus, both $A - B$ and $B - A$ are non-empty. By PD, there exists $j \in (A - B) \cup (B - A)$ such that: (i) $(A - \{k\})_{j\to k} \succ (B - \{k\})_{j\to k}$; (ii) $(B - \{k\})_{j\to k} \cup \{j\} \succ (A - \{k\})_{j\to k} \cup \{j\}$. By definition of k, $j > k$. If $j \in A$, then $(A - \{k\})_{j\to k} = A - \{j\}$ and $(B - \{k\})_{j\to k} = B - \{k\}$, implying that (i) contradicts the inductive step. It follows that $j \in B$. Therefore, (ii) can be written as $\{k\} \cup B \succ (A - \{k\}) \cup \{j\}$ and $B \succ A$ can be written as $\{j\} \cup B \succ (A - \{k\}) \cup \{k\}$. By CI(1), $B \succ A - \{k\}$, contradicting the inductive step. This establishes that $\succ$ is lex (max-max, lex-max-min).

Finally, let us verify that if $\succ$ is lex (max-max, lex-max-min), then axioms CI(1), D, PD and BET are satisfied. The last three can be shown to be satisfied (note that PD is satisfied because $\succ$ satisfies independence). Let us prove that CI(1) is also satisfied. As shown in the proof of Theorem 1, if $\max(A \cup C) = \max(B \cup D)$ and $\max(A \cup D) = \max(B \cup C)$, with at least one strict inequality, then $\max(A) > \max(B)$. If $\max(A \cup C) = \max(B \cup D) = k$ and $\max(A \cup D) = \max(B \cup C) = k'$, then $k = k'$ and $\max(A) = \max(B) = k$. Hence, if $A \cup C \succ B \cup D$ and $A \cup D \succ B \cup C$, then it must be according to the lex-max-min rule, which is inversely symmetric to the lexicographic rule. Thus, the proof that $A \succ B$ is the same as the proof of the equivalent claim in Theorem 1. $\square$

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NOTES

1. The class of max-max (weak) orders has been given several derivations in the literature (e.g., Barbera and Pattanaik, 1984). All the derivations I am aware of are independent of the present one and are based on variants on independence and the Gärdenfors Principle (Gärdenfors, 1976). An order $\succ$ satisfies the Gärdenfors Principle if for every non-empty set $A \subset X$ and every element $k \in X - A$, if $\{k\} \succ \{a\}$ for all $a \in A$ then $A \cup \{k\} \succ A$, and if $\{a\} \succ \{k\}$ for all $a \in A$ then $A \succ A \cup \{k\}$.
2. Cases 2 and 3 are less interesting. In case 3, cross inference is not valid even without prior knowledge about the ordering, let alone when such knowledge exists. In case 2, Proposition 3.2 shows that CI(2) is equivalent to independence, with prior knowledge or without it.
3. Bossert et al. (2000) study similarly named, yet different decision rules, in which sets are compared according to their best and worst elements, and if two sets have the same best and worst elements, they are compared according to their second-best and second-worst elements.

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